CS 61B Fall 2026

Asymptotics II

Discussion 06: October 5, 2026

1 Asymptotics II Summary

Asymptotic Notation Recap. Given a function $R(N)$:

- $R(N) \in \Theta(f(N))$: This means $R(N)$ grows at the **same rate** as $f(N)$.
- $R(N) \in O(f(N))$: This means $R(N)$ grows **no faster than** $f(N)$. Used as an **upper bound**.
- $R(N) \in \Omega(f(N))$: This means $R(N)$ grows **no slower than** $f(N)$. Used as a **lower bound**.

Informally, we can think of Θ as $=$, O as $\leq$, and Ω as $\geq$. Suppose $R(N) = N^3 + 2N + \cos(N)$, then all of the following are true:

- $R(N) \in \Theta(N^3)$
- $R(N) \in O(N^3)$
- $R(N) \in O(N^{1271})$
- $R(N) \in \Omega(N^2)$
- $R(N) \in \Omega(1)$

If the asymptotic runtime depends on the input, we **cannot** give a Θ bound for the overall runtime of the code. For example, suppose we want to characterize the runtime $R(N)$ of a function `contains` that searches an unsorted array to see if some item is present, where N is the length of the array. All four of the following statements are true:

- The worst case runtime of `contains` is $\Theta(N)$.
- The runtime of `contains` is $O(N)$.
- The best case runtime of `contains` is $\Theta(1)$.
- The runtime of `contains` is $\Omega(1)$.

Useful Sum Formulas.

$$\text{Arithmetic:} \qquad 1 + 2 + 3 + \cdots + Q = \frac{Q(Q+1)}{2} \in \Theta(Q^2)$$

$$\text{Geometric:} \qquad 1 + 2 + 4 + 8 + \cdots + Q = 2Q - 1 \in \Theta(Q)$$

Analyzing Recursive Functions (Tree Method). To find the runtime of a recursive function:

(a) **Draw the call tree.** Each node represents one call; label it with the non-recursive work done in that call (e.g. the cost of loops, helper calls).

(b) **Sum each layer.** Add up the work across all nodes at the same depth.

(c) **Count the layers.** Determine the height of the tree (how many times can N be divided/reduced before hitting the base case?).

(d) **Total it up.** Total work = (work per layer) $\times$ (number of layers), or sum the per-layer costs if they vary.

Note, non-61B sources are likely to write the geometric sum as shown below. If you like this equivalent formulation better, it's fine to use this instead:

$$\text{Geometric:} \qquad 1 + 2 + 4 + \cdots + 2^k = 2^{k+1} - 1 \in \Theta\left(2^k\right)$$

2 Asymptotics of Weighted Quick Union

Note: for all big Ω and big O bounds, give the *tightest* bound possible.

(a) Suppose we have a Weighted Quick Union (WQU) without path compression with N elements.

1. What is the runtime, in big Ω and big O, of `isConnected`?

Lower bound: $\Omega($ 1 $)$	Upper bound: $O($ $\log N$ $)$

2. What is the runtime, in big Ω and big O, of `connect`?

Lower bound: $\Omega($ 1 $)$	Upper bound: $O($ $\log N$ $)$

In the best-case, if we're checking if `a` and `b` are connected, `a` is the root, and `b` is a node directly below the root. This means we only have to traverse one edge of the tree, which is constant time. In the worst-case, we have to traverse the entire height of the tree, and Weighted Quick Union gives us a worst-case height of $\log N$, hence the upper-bound of $O(\log N)$. Similar logic applies to the `connect` method.

(b) Suppose we add the method `addToWQU` to a WQU without path compression. The method takes in a list of `elements` and `connects` them in a random order, stopping when all elements are connected. Assume that all the `elements` are disconnected before the method call.

```
void addToWQU(int[] elements) {
    int[][] pairs = pairs(elements);
    for (int[] pair: pairs) {
        if (size() == elements.length) {
            return;
        }
        connect(pair[0], pair[1]);
    }
}
```

The `pairs` method takes in a list of `elements` and generates all possible pairs of elements in a random order. For example, `pairs([1, 2, 3])` might return `[[1, 3], [2, 3], [1, 2]]` or `[[1, 2], [1, 3], [2, 3]]`.

The `size` method calculates the size of the largest component in the WQU.

Assume that `pairs` and `size` run in constant time. NOTE: PAIRS CANNOT POSSIBLY BE CONSTANT TIME.

What is the runtime of `addToWQU` in big Ω and big O?

Lower bound: $\Omega($ N $)$	Upper bound: $O($ $N^2 \log N$ $)$

Hint: Consider the number of calls to **`connect`** *in the best case and worst case. Then, consider the best/ worst case time complexity for one call to* **`connect`***.*

Note that the if-statement terminates the method when the disjoint set becomes fully connected. The best case occurs when there is a sequence of pairs such that each `connect()` operation takes constant time and the tree becomes connected as quickly as possible. This will happen if we have a sequence $(0, 1), (0, 2), ..., (0, N-1)$, which consists of $N-1$ operations each taking constant time (ie. the best case for `connect` from part a). Note that long running-times occur when an element (e.g. 0) is not connected for many operations, and in the worst-case, 0 is not connected until the last N operations. This results in a tree of height $\log N$ and requires up to $N^2 - N + 1$ iterations.

(c) Let us define a **matching size connection** as `connecting` two distinct components in a WQU of equal size. For instance, suppose we have two trees, one with values 1 and 2, and another with the values 3 and 4. Calling `connect(1, 4)` is a matching size connection since both trees have 2 elements.

What is the **minimum** and **maximum** number of matching size connections that can occur after executing `addToWQU`? Assume `N`, i.e. `elements.length`, is a power of two. Your answers should be exact.

minimum: `1`, maximum: `N - 1`
The minimum number occurs for the sequence above, where there is only one matching size connection: (0, 1). The maximum number is a bit more tricky, but occurs if we pairwise-connect the elements together, then pairwise connect those, and so on. An example for N=8 elements is as follows: (0, 1), (2, 3), (4, 5), (6, 7), (0, 2), (4, 6), (0, 4). In general, there are $\frac{N}{2}$ matching-size connections of size 1, $\frac{N}{4}$ matching-size connections of size 2, and so on, up until one matching-size connection of size $\frac{N}{2}$. This is the sum $\frac{N}{2} + \frac{N}{4} + \frac{N}{8} + ... + 2 + 1$, which simplifies to $N - 1$.

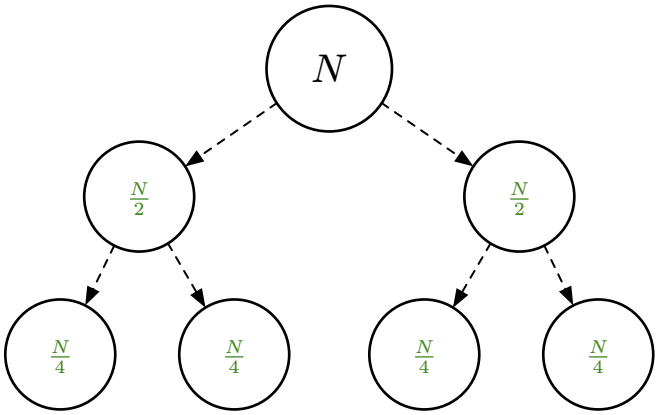
3 Forestry

When analyzing the runtime of recursive functions, it's almost always a good idea to map out the recursive calls using a visual diagram. Consider the function below, where `blocky(int M)` is a static method that runs in $\Theta(M)$

```
public static void blobby(int N) {
  if (N <= 1) { return; }
  blocky(N);
  blobby(N / 2);
  blobby(N / 2);
}
```

It looks a bit complicated to analyze! Let's break it down.

(a) Fill out **each node** of the recursive call tree below with the amount of work done relative to N on each respective call of `blobby`. The first call has been filled in for you. Then, sum up the work across each layer, and try to predict a rule for the amount of work per layer.



Amount of work on layer 0:

N

Amount of work on layer 1:

N

Amount of work on layer 2:

N

⋮

Amount of work on layer l:

N

(b) Now, we need to figure out how many layers the tree has. This is equivalent to asking the question *"how many times can N be divided by 2 before it hits the base case of 1?"*

$l =$

$\log N$

. *Hint: l is a function of N.*

(c) Now, add up the work on all l layers.

There are

$\log N$

layers, and, since

N

work is done on each layer,

the total runtime is equal to

$\log N$

*

N

, which is in $\Theta(N \log N)$.

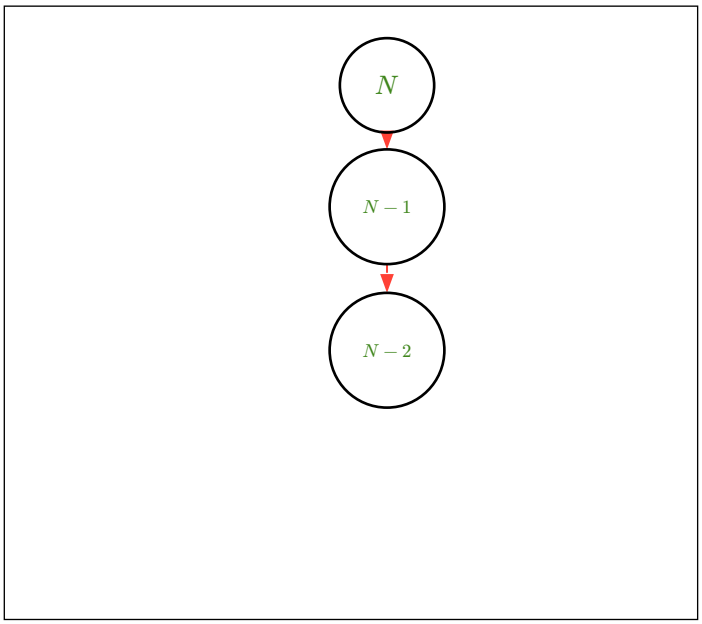
4 Forestry II

Drawing tree diagrams is applicable to a wide variety of asymptotics problems! Let's try another. Once again, `blocky(int M)` is a static method that runs in $\Theta(M)$.

```
public static int blobbier(int N) {
  if (N <= 0) { return 0; }
  blocky(N);
  return 17 + blobbier(N - 1);
}
```

(a) Draw a recursive call tree in the box below, starting with the initial call. Then, analyze the amount of work across each layer.

Hint: How much work is done in layer 0: 1, l, or N?



Amount of work on layer 0:

N

Amount of work on layer 1:

$N-1$

Amount of work on layer 2:

$N-2$

⋮

Amount of work on layer l:

$N-l$

(b) Now, how many layers does the tree have?

$l =$

N

(c) Now, add up the work on all l layers, and theta-bound it.

What is the final runtime?

Hint: Remember the sum formulas.

This is an arithmetic sum! Writing out the sum of the work from the topmost node to the bottom, we get

$N + (N-1) + (N-2) + ... + 1$

$= 1 + 2 + ... + N$

$= \Theta(N^2)$

Runtime: $\Theta\left(N^2\right)$

5 The Re-Cursed Swamp

Sometimes, it isn't possible to give a theta bound for an entire function. In these cases, it's best to analyze the best and worst-case inputs and evaluate the runtime on those to produce a "tightest" upper and lower bound.

(a) Consider the function below…

```
public static int curse(int N) {
    if (N % 2 == 0 || N <= 0) {
        return 0;
    } else {
      for (int i = 0; i < N; i++) {
        System.out.println("You have been cursed!");
      }
      return curse(N - 2);
    }
}
```

What type of input will result in the best-case runtime? What type of input will result in the worst-case runtime?

The best-case input is an even number. In this case, the top conditional will immediately cause the function to return, resulting in a $\Omega(1)$ lower bound.

The worst-case input is an odd number. In this case, the top conditional will never cause the function to return, resulting in an $O(N^2)$ upper bound.

Give a tight Ω and O bound that correspond to the lower and upper bounds on this function's runtime. That is, don't just say sometihng like $O(2^N)$ which is technically true, but useless.

Feel free to use the tree-drawing technique from questions 2 and 3!

Lower bound: $\Omega($ 1 $)$	Upper bound: $O($ N^2 $)$

(b) Give the tightest runtime bound(s) for the function below. We can assume the `System.arraycopy` method takes $\Theta(N)$ time, where N is the number of elements copied. The official signature is `System.arraycopy(Object sourceArr, int srcPos, Object dest, int destPos, int length)`. Here, `srcPos` and `destPos` are the starting points in the source and destination arrays to start copying and pasting in, respectively, and `length` is the number of elements copied.

```
public static void silly(int[] arr) {
    if (arr.length <= 1) {
      return;
    }

    int newLen = arr.length / 2;
    int[] firstHalf = new int[newLen];
    int[] secondHalf = new int[newLen];

    System.arraycopy(arr, 0, firstHalf, 0, newLen);
    System.arraycopy(arr, newLen, secondHalf, 0, newLen);

    silly(firstHalf);
    silly(secondHalf);
}
```

$\Theta(N)$

At each level, we do N work, because the call to `System.arraycopy`. You can see that at the top level, this is N work. At the next level, we make two calls that each operate on arrays of length N/2, but that total work sums up to N. On the level after that, in four separate recursive function frames we'll call `System.arraycopy` on arrays of length N/4, which again sums up to N for that whole layer of recursive calls.

Now we look for the height of our recursive tree. Each time, we halve the length of N, which means that the length of the array N on recursive level k is roughly $N * \left(\frac{1}{2}\right)^k$. Then we will finally reach our base case $N \leq 1$ when we have $N * \left(\frac{1}{2}\right)^k = 1$. Doing some math, we see this can be transformed into $N = 2^k$, which means $k = \log_2(N)$. In other words, the number of layers in our recursive tree is $\log_2(N)$. If we have $\log_2(N)$ layers with $\Theta(N)$ work on each layer, we must have $\Theta(N \log(N))$ runtime.

(c) Given that `exponentialWork` runs in $\Theta(3^N)$ time with respect to input N, give the tightest runtime bound(s) for `yellowWood`.

Hint: This one is hard! Drawing trees will be of utmost importance. If you suspect that the runtime cannot be theta-bounded, drawing one for the best case and one for the worst case can be a good idea.

```
public void yellowWood(int N) {
  if (Math.random() > 0.9) {
    twoPathsDiverge(N, 2);
  } else {
    twoPathsDiverge(N, 1);
  }
}

private void twoPathsDiverge(int N, int j) {
  if (N <= 1) {
      return;
  }
  exponentialWork(N);
  for (int i = 0; i < 3; i++) {
    twoPathsDiverge(N - j, j);
  }
}
```

$\Omega(3^N)$, $O(N * 3^N)$

For the best case, the first level has $\Theta(3^N)$ work, the next level has $\Theta(3^{N-1})$ work, and so on until the last level which has approximately $\Theta\left(3^{\frac{N}{2}}\right)$ work. This gives the sum $\Theta(3^N) + \Theta(3^{N-1}) + ... + \Theta\left(3^{\frac{N}{2}}\right) = \Theta(3^N)$.

For the worst case, each level has $\Theta(3^N)$ work. Given that there are $\sim N$ layers, summing up the tree results in $O(N * 3^N)$ for an upper bound.